Enhanced synchronizability via age-based coupling

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In this Brief Report, we study the synchronization of growing scale-free networks. An asymmetrical age-based coupling method is proposed with only one free parameter α. Although the coupling matrix is asymmetric, our coupling method could guarantee that all the eigenvalues are non-negative reals. The eigenratio R will approach 1 in the large limit of α.

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One of the main goals in the study of network science is to understand the relation between the network structure and the dynamical processes performed upon it [1,2]. A typical collective dynamic on the networked system is synchronization, where all the participants behave alike, even exactly the same. This phenomenon exists everywhere from physics to biology [3] and has been observed for hundreds of years. With the partial knowledge of relations between network structure and its synchronizability [4–8], scientists have proposed many methods to enhance the network synchronizability [9–18]. Generally speaking, these methods can be divided into two classes: the modification of the network structure [9–11] and the regulation of the coupling pattern [12–18]. In the former class, networks are modified either to shorten the average distance [10] or to eliminate the maximal betweenness [9,11]. In the latter case, the network structure is kept unchanged, while the coupling matrix is elaborately designed (often asymmetrically) to improve the synchronizability [12–18].

The first coupling pattern other than the symmetric case was proposed by Motter, Zhou, and Kurths (MZK) [12–14], in which the coupling strength a node i receives from its neighbors is inverse to k_i^{β} with k_i the degree of i. The coupling pattern can sharply enhance the network synchronizability, with $\beta=1$ the optimal case. After this pioneering work, many coupling patterns [15–18] have been presented to further enhance the network synchronizability. In Ref. [15], Hwang *et al.* presented a coupling method taking into account the age of nodes, which makes the network even more synchronizable than the optimal case of the MZK coupling pattern. In this pattern, each node receives coupling signals from its neighbors, with each receiving coupling strength taking one of the two values: if the neighbor is older, the coupling strength takes the larger value, otherwise it takes the smaller one. To separate the different coupling situations (i.e., from older to younger and from younger to older) by using two discrete coupling strengths is the simplest way one can imagine. However, since each node has its own age, a coupling method taking into account the age difference between each pair of coupled nodes may further enhance the synchronizability. Moreover, the coupling matrix in Ref. [15] has complex eigenvalues, leading to a complicated analysis. An elaborately designed method, as shown in this Brief Report, could guarantee that all the eigenvalues are nonnegative reals, thus one can easily predict the synchronizability of the underlying network by considering the real eigenratio only. This method is analogous to the modified MZK method introduced in Ref. [14], which further enhances synchronization without involving any complex eigenvalue. However, in contrast to the modified MZK method [14], our model is based on the ages rather than degrees of the nodes.

In a dynamical network, each node represents an oscillator and the edges represent the couplings between nodes. For a network of N linearly coupled identical oscillators, the dynamical equation of each oscillator can be written as

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}^j), \quad i = 1,2,\ldots,N, \tag{1}$$

where $\dot{\mathbf{x}}^i=\mathbf{F}(\mathbf{x}^i)$ governs the essential dynamics of the ith oscillator, $\mathbf{H}(\mathbf{x}^j)$ the output function, σ the coupling strength, and G_{ij} an element of the $N\times N$ coupling matrix G. To guarantee the synchronization manifold an invariant manifold, the matrix G should have zero row-sum. The collective dynamic starts from a disorder initial configuration, under suitable conditions, the couplings will make the oscillators' states nearer and nearer. Eventually, all the individuals oscillate together and a synchronization phenomenon emerges.

In the simplest symmetric way, the coupling matrix G has the same form as the Laplacian matrix L, that is, $G_{ij}=L_{ij}$, where

$$L_{ij} = \begin{cases} k_i & \text{for } i=j, \\ -1 & \text{for } j \in \Lambda_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Here Λ_i is the set of i's neighbors. Because of the symmetry and the positive semidefinite of L, all its eigenvalues are nonnegative reals and the smallest eigenvalue λ_0 is always zero, for the rows of L have zero sum. And if the network is connected, there is only one zero eigenvalue. Thus, the eigenvalues can be ranked as $\lambda_0<\lambda_1\le\lambda_2\le\cdots\le\lambda_{N-1}$. When the stability zone is bounded, according to the criteria of the

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master stability function [19,20] (see also the unbounded case [21,22]), the network synchronizability can be measured by the eigenratio $R=\lambda_{N-1}/\lambda_1$: The smaller it is the better the network synchronizability and vice versa.

The couplings between nodes are not limited to the symmetric mode, however, generally, the eigenratio of an asymmetric coupling matrix is complex (e.g., the eigenratio in Ref. [15]). Therefore, in order to ensure that the network has strong synchronizability, not only should the ratio of the real part be taken into account, but also the imaginary part must be guaranteed to be as small as possible. In Ref. [15], the simulation result indicated that although the ratio of the real part is the smallest, at the same time the imaginary part is the largest. To overcome this blemish and give further enhancement of synchronizability, we bring forward a coupling pattern in which the coupling strength between two connected nodes is the function of their age difference. The age of node i is signed by the time it enters into the network, thus smaller i corresponds to older age. The coupling matrix proposed here is

$$G_{ij}=\begin{cases}1 & \text{for } i=j,\\ -\dfrac{e^{-\alpha(j-i)/N}}{S_i} & \text{for } j\in\Lambda_i,\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

where $S_i=\sum_{j\in\Lambda_i}e^{-\alpha(j-i)/N}$ is the normalization factor. In this coupling pattern, the case of $\alpha=0$ degenerates to the optimal case of MZK coupling pattern. When $\alpha>0$, the old nodes have stronger influence than the younger ones; while for $\alpha<0$, younger nodes are more influential.

It can be proved that although the coupling between nodes is asymmetric, all the eigenvalues of matrix G are reals. Note that the coupling matrix defined in Eq. (3) can be written as

$$G=DL', \tag{4}$$

where

$$D=\mathrm{diag}(e^{2\alpha}/S_1,e^{4\alpha}/S_2,e^{6\alpha}/S_3,\ldots,e^{2N\alpha}/S_N) \tag{5}$$

is a diagonal matrix and $L'=(L'_{ij})$ is a symmetric zero row-sum matrix, whose off-diagonal elements are

$$L'_{ij}=-e^{-\alpha i}e^{-\alpha j}. \tag{6}$$

From the identity [13]

$$\det(DL'-\lambda I)=\det(D^{1/2}L'D^{1/2}-\lambda I) \tag{7}$$

valid for any λ, we have that the spectrum of eigenvalues of matrix G is equal to the spectrum of a symmetric matrix defined as

$$H=D^{1/2}L'D^{1/2}. \tag{8}$$

As a result, although the coupling matrix G is asymmetric, the eigenvalues of matrix G are all nonnegative reals and the smallest eigenvalue is always zero. Therefore, in contrast to the complicated case in Ref. [15], the synchronizability based on the present coupling pattern can be measured directly by the real eigenratio R.

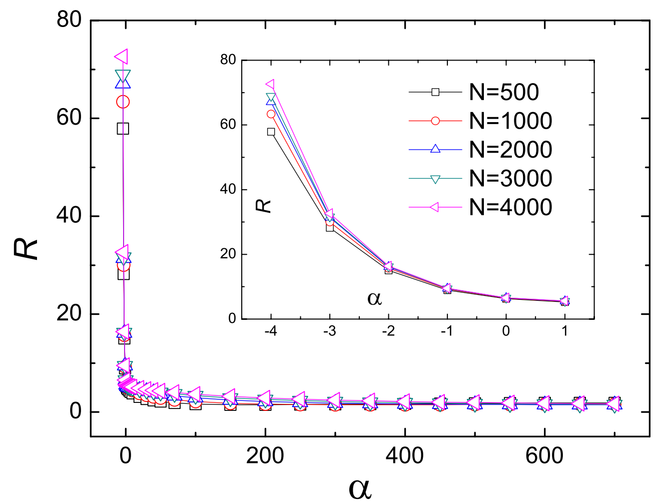

FIG. 1. (Color online) The eigenratio R vs α in BA networks with average degree $\langle k\rangle=6$. The inset displays the details for the interval $\alpha\in[-4,1]$. Each data point is obtained by averaging over 50 different network configurations. The eigenratio R goes to 1 in the large limit of α.

In Fig. 1, we report the changes of eigenratio R with the parameter α in BA networks [23] at different sizes. One can easily conclude from Fig. 1 that with the increase of α the eigenratio decreases sharply, no matter what the network size. It is shown that in growing networks, if the couplings from older nodes are stronger than the reverse, the network will get better synchronizability. Otherwise, if the coupling from younger to older ones is strengthened (see the cases of $\alpha<0$ in the inset), the system becomes very hard to synchronize. When α goes to infinity, the eigenratio will converge to 1, which is the smallest eigenratio corresponding to the best synchronizability [24]. Actually, in the case $\alpha\to+\infty$, each node is coupled by its oldest neighbor, while the oldest node in the network is uncoupled. Thus, the coupling matrix (whose rows are sorted by the descending order of ages) becomes a lower triangular matrix with all the diagonal elements are 1 except the first one G_{11}, being equal to zero. Therefore, all the nonzero eigenvalues are 1.

Although there is a method to design a coupling pattern having optimal synchronizability (i.e., $R=1$) [24], for growing networks, using the age of each node is a simple and feasible way since to know any other measures of nodes may cost much for huge-size system, and this age-based coupling can guarantee the connectivity of the whole network. Mathematically speaking, the synchronizability here is a measure on the stability of invariant synchronization manifold. We call a synchronization manifold is stable if the dynamical system can automatically return to this manifold after a perturbation. A network G has better synchronizability than another network G' means that any collective dynamics with identical oscillators upon G' having a stable synchronization manifold must have a stable synchronization manifold for G, while there exists certain dynamics having stable synchronization manifold for G, but not for G'. However, better synchronizability does not guarantee a shorter converging time from disorder initial configuration to synchronized state. Actually, Nishikawa and Motter [24] found that the synchronizing process may take longer in the optimal network with $R=1$ (see also a similar conclusion for nonidentical oscillators [25]). Based on the current coupling pattern, one can obtain an acceptable trade-off between synchronizability and converging time by tuning the parameter α. Moreover, comparing with the pioneer work by Hwang *et al.* [15], our coupling method can achieve even smaller R, and does not need to deal with the complicated and tedious analysis on complex eigenratios. Our elaborately designed coupling pattern can guarantee the eigenratio a real number, just as in the degree-based models of Refs. [14,18].

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